

APPLICATIONS OF GROUPS AND ISOMORPHIC GROUPS TO TOPICS IN THE STANDARD CURRICULUM, GRADES 9-11: PART I

Many relationships between groups and topics of secondary school mathematics are shown by the author, who proposes that the study of groups be included as standard fare in the mathematics curriculum of the average college-bound student.

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THE subfield of pure mathematics that has grown most significantly in the past few decades is that of algebra, by which is meant "higher" or "abstract" algebra and linear algebra. Twenty years ago courses in algebra were at the advanced undergraduate and graduate level, and it was easy to become a certified mathematics teacher without having any knowledge of groups, rings, fields, or vector spaces. Today virtually all prospective teachers take a course in which some of these structures are studied.

Yet, if we judge the situation from our textbooks, we find most students who take eleven years of school mathematics rarely explicitly encounter any of the common algebraic structures except by way of discussions of field properties. Even though such material often appears early in texts and is seemingly necessary for future work, the ideas are seldom used in any constructive way.

At the present time, in the United States, only books written for *bright* students have used structures as an integral part of the development. Two examples are *A Vector Approach to Euclidean Geometry*, by Vaughan and Szabo (New York: Macmillan, 1972) and *Unified Mathematics*, vols. 1-3, by Fehr, Fey, and Hill (Reading, Mass.: Addison-

Wesley Publishing Co., 1972). Yet in foreign texts, groups and other structures are not uncommon even for average students. For example, see the texts of the School Mathematics Project (New York: Cuisenaire Co.) and *Modern Mathematics 1 and 2*, by Papy (New York: Macmillan Co., 1968, 1970).

The United States situation, in view of the importance that algebra now has in mathematics, may be serious. A recent survey found that 15% of the doctorates in mathematics in 1968-71 were in algebra, second only to analysis (21%) among the branches (*CBMS Newsletter*, vol. 7, no. 3, May 1972). However, it cannot be assumed that something ought to be studied by high school students just because it is important to mathematicians. But it can be assumed that algebraic structures should at least be given serious consideration. "Groups" are among the most fundamental of these structures and seem to be very amenable to early study by students. For example, several books by Dienes and Golding (1967 [a], [b], and [c]) contain activities for children in grades 3 through 6.

The major purpose of this article is to exhibit relationships between groups and various topics in secondary school mathematics. By presenting a wide variety of topics and approaches, it is hoped to add evidence to the increasingly strong case for the inclusion of some study using groups as standard fare in the mathe-

matics curriculum of the average college-bound student. (Part II of this article will appear in next month's issue.)

Preliminary Remarks

The author has experimented with various ways of introducing groups to tenth and eleventh graders. For average students it seems best not to begin with a definition and examples. A more natural approach is to begin with a variety of situations in which the group properties appear but the groups are not identified and to use the identification of groups as a device to summarize and unify previously unjoined content. Similarly, it seems better to present numerous situations involving isomorphic groups before either the terminology or any formal definitions are given. Approaches using these strategies may be found in materials currently being tested (Usiskin 1972), and some similar ideas have been used by Dienes at the elementary level (Dienes and Golding 1967 [a], [b], and [c]). However, in this article, some definitions and examples are given before any applications are given to make it easier to identify specifically the mathematics involved in each application.

Here is a rather standard definition of "group."

DEFINITION: Let $*$ be a binary operation on a set S . Then S and $*$ constitute the group $\langle S, * \rangle$ if and only if

1. $*$ is closed in S
2. $*$ is associative in S
3. there is an identity for $*$ in S
4. each element in S has an inverse under $*$.

The properties of closure, associativity, the existence of an identity, and the existence of inverses are called *group properties* here. It is notable that commutativity is not a group property.

Students in grades 10 and 11 seem to find it rather easy to check whether a set and an operation form a group, since the four properties have almost always been

taught to them some time or another by the end of the ninth grade.

Here are some notable groups involving some or all of the real numbers. Groups 1-3 are *additive groups* because the operation (denoted by $+$) is addition. Groups 4-9 are *multiplicative groups* (" \cdot " stands for multiplication). That the operation is just as important as the set will be seen from some of the applications.

1. $\langle \text{set of integers}, + \rangle$
2. $\langle \text{set of rational numbers}, + \rangle$
3. $\langle \text{set of real numbers}, + \rangle$
4. $\langle \text{set of positive rationals}, \cdot \rangle$
5. $\langle \text{set of nonzero rationals}, \cdot \rangle$
6. $\langle \text{set of positive real numbers}, \cdot \rangle$
7. $\langle \text{set of nonzero real numbers}, \cdot \rangle$
8. $\langle \{0\}, \cdot \rangle$
9. $\langle \{1, -1\}, \cdot \rangle$

Group 8 is notable in that the multiplicative identity for this group is 0. Group 8 is also the only possible multiplicative group whose set contains the real number 0. Groups 8 and 9 are *finite groups*; the other groups are *infinite*.

Isomorphic groups are best introduced by example. Here we list the integers and the integral powers of 2. The listing is arranged in such a way that m and 2^m are on the same horizontal line.

<u>m</u>	<u>2^m</u>
0	1
1	2
-1	.5
2	4
-2	.25
③	⑧
-3	.125
4	16
④	①.0625
5	32
-5	.03125
⋮	⋮
⋮	⋮
Add	Multiply

Two horizontal rows have been identified by drawing circles around the elements. Now, using the two marked rows,

we *add* the elements in the left column and *multiply* the elements in the right column.

$$3 + -4 = -1 \quad (8)(.0625) = .5$$

The answers (-1 at left, .5 at right) lie in the same row—the third row from the top. This is a characteristic of isomorphic groups—the answers correspond.

DEFINITION: *Two groups $\langle S, * \rangle$ and $\langle T, \# \rangle$ are isomorphic if and only if there is a one-to-one correspondence between the elements of S and T so that if a in S corresponds to x in T and b in S corresponds to y in T then $a * b$ corresponds to $x \# y$.*

In this example, S is the set of integers and T is the set of integral powers of 2. $*$ is addition and $\#$ is multiplication. The one-to-one correspondence is given by the rule

$$m \rightarrow 2^m.$$

That is, the one-to-one correspondence is the one-to-one function f , with $f(m) = 2^m$.

How do we know that $m \rightarrow 2^m$ is an isomorphism? This is easy to prove using the fundamental property of exponents.

Given

$$m \rightarrow 2^m \quad \text{for any integer } m,$$

then

$$n \rightarrow 2^n$$

and certainly

$$m + n \rightarrow 2^{m+n}.$$

But we know that

$$2^{m+n} = 2^m \cdot 2^n,$$

so

$$m + n \rightarrow 2^m \cdot 2^n.$$

Thus, adding the integers and multiplying the corresponding powers of 2 yield corresponding answers. The correspondence is one-to-one because different integers give rise to different integral powers and vice versa.

There seem to be as many applications of isomorphic groups in elementary mathematics as there are of groups themselves.

In particular, the isomorphism $m \rightarrow 2^m$ is of a type applied in Part II of this article.

Some properties of isomorphic groups are worth noting. (The reader should refer back to the columns of integers and powers.) In isomorphic groups, the identities correspond (0 and 1 in the top row). Inverses in one group correspond to inverses in the other group. (For example, 5 and -5 in the additive group of integers correspond to 32 and .03125 in the isomorphic multiplicative group of powers of 2.) And if one set and operation form a group and there is a one-to-one correspondence with another set and operation and answers correspond, then the second set and operation form a group.

Groups and Sentence Solving

The applications of groups to sentence solving are well known, found in some high school texts, yet important enough to deserve repeating here.

Essentially the idea is this: A set S and an operation $*$ constitute a group exactly when there is a *unique* solution in S to each of the equations

$$a * x = b \quad y * a = b$$

regardless of the choice of a and b from S .

Application 1: It is exactly the group properties that are necessary to show that the equation $a + x = b$ has a unique solution when a and b are arbitrarily selected from a set S and x is to be in S .

Given

$$a + x = b$$

Existence of inverses in S

Well-definedness of $+$

$$-a + (a + x) = -a + b$$

Associativity of $+$

$$(-a + a) + x = -a + b$$

Property of $-a$

$$0 + x = -a + b$$

0 is identity for $+$

$$x = -a + b$$

And $-a + b$ is in S because of closure, the fourth defining property of $(S, +)$.

In particular, the equation $a + x = b$ cannot always be solved in the set of positive integers, because the positive integers do not form an additive group. For example, $3 + x = 2$ has no solution for most third graders, because they have not yet studied negative numbers. In the set of all integers, the equation always has a solution. That is, without calculating the solution, we can be sure that $-4,829 + x = 64,710$ has a solution that is an integer. And given a and b rational, no irrational could ever be a solution to $a + x = b$.

Application 2: The nonuniqueness of solutions to certain simple equations involving multiplication can be explained using groups.

In the set of real numbers, if one attempts to solve the equation $ax = b$ in the same way that $a + x = b$ was solved, the process breaks down at the second line, for not all reals have inverses under multiplication. This forces one to deal with equations of the form $ax = b$ by considering cases.

When a and b are *nonzero* reals, there is always a unique solution to the sentence $ax = b$. (In solving this equation, one is working in the multiplicative group of nonzero reals.)

There are only three equation forms that are not covered by this uniqueness. They are these:

$$0 \cdot x = 0$$

$$0 \cdot x = b, b \neq 0$$

$$a \cdot x = 0, a \neq 0$$

Each of these is an important example of more general types; the first is an example of an equation with infinitely many solutions, the next an example of an equation with no solution, and the last the simplest instance of the product of numbers equaling zero, forcing one of them to be zero. The use of group properties here explains why there are no other special cases.

Application 3: By using groups, the solving of linear systems can be related to the solution of simple equations.

We restrict our attention to systems of the form

$$ax + by = e$$

$$cx + dy = f$$

because these are most commonly encountered. (Larger systems could be similarly treated.) In matrix language, this system is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

A group that can be involved here is the multiplicative group of invertible 2×2 matrices. A simple equation in this group might have the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} e & g \\ f & h \end{bmatrix}.$$

Multiplying the matrices gives the original system and a second system with coefficients $a, b, c,$ and d to be solved for z and w . This gives the student the correct realization that with matrix methods one could solve two systems of two equations with virtually the same amount of work required to solve one system. We think of this as an equation of the form

$$A \cdot X = B$$

and solve it just like the earlier $ax = b$. The parallel occurs in that there is a unique solution if and only if A has an inverse. It is well known that this is true exactly when $ad - bc \neq 0$. In other words, there is only the possibility of nonunique solutions when $ad - bc \neq 0$. In traditional language, we get inconsistent or dependent equations only when $ad - bc = 0$. This compares with the nonunique solutions of $ax = b$ when $a = 0$.

Groups and Number Operations

It is very useful to have geometric pictures for algebraic or arithmetic opera-

tions. Such pictures can be used to check answers, verify properties, and in some cases lead to new results. In certain instances, the pictures can be interpreted as representing a second group isomorphic to a group from algebra or arithmetic.

Application 4: Addition of real numbers is isomorphic to composition of one-dimensional translations (slides).

This application is unique because the geometry is so easy. In fact, the slides are so much easier than addition of real numbers that we teach children to add by using the slides as a model. (Compare this with the use of rotations for modular or clock arithmetic addition.) Figure 1 is a picture of $2 + -3.2 = -1.2$.

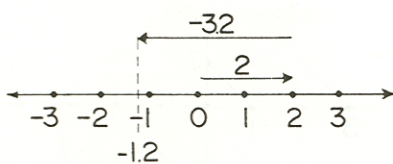


Fig. 1

The isomorphism can be seen by corresponding the elements as in figure 2. It is clear that there is a one-to-one correspondence and that the answers correspond.

Another way of showing the relationship between addition and slides is to pick a set of numbers and add a fixed number to each of them. When graphed, the result is as if the graphs of the real numbers had been slid. The example

<i>Real number</i>		<i>Slide</i>
2	corresponds to	2 units to the right
added to		composed with
-3.2		3.2 units to the left
<hr/>		<hr/>
Answers: -1.2		1.2 units to the left

Fig. 2

shows the set $\{x: 1 \leq x \leq 2\}$ to which 5.5 has been added (fig. 3).

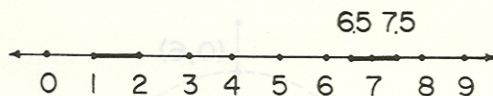


Fig. 3

Application 5: Multiplication of real numbers is isomorphic to composition of size transformations. (The size transformation or dilatation of magnitude k and center $(0, 0)$ maps (x, y) onto (kx, ky) . This transformation is fundamental in the study of similarity.)

Either one-dimensional or two-dimensional size transformations will work here; figure 3 shows a two-dimensional picture of "multiplying by 3." The idea is to coordinatize the plane and multiply the coordinates of all points of the original picture by 3 to get the image picture.

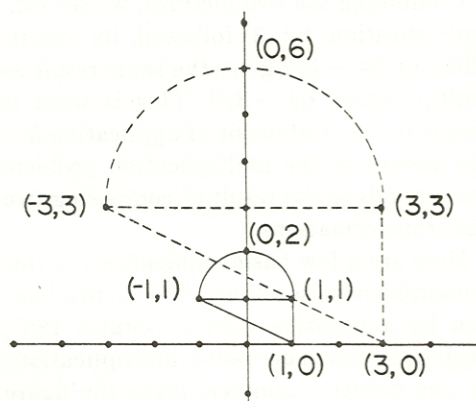


Fig. 4

Multiplying by $-1/2$ not only gives an image whose linear dimensions are $1/2$

those of the original, but also rotates the drawing 180° .

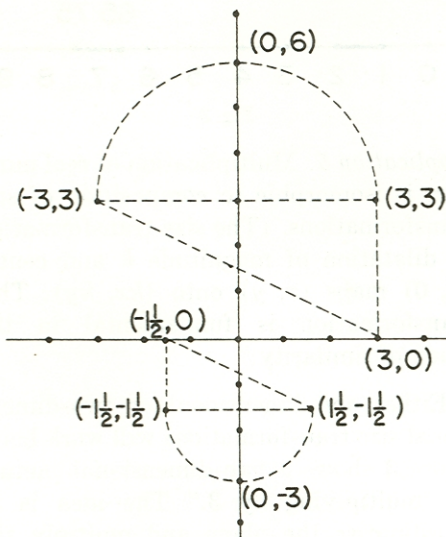


Fig. 5

Combining the two pictures, we see that multiplication by 3 followed by multiplication by $-1/2$ gives the same result as multiplication by $-3/2$. This is what is meant by the statement of application 5—the answer to the multiplication problem corresponds to the result of composing two size transformations.

Here are a few basic applications of this isomorphism in teaching: first, multiplication by a negative number rotates 180° , suggesting that successive multiplications by two negative numbers leave the figure “right side up.” This is a reasonable model of the product of two negative numbers being positive. Second, if we take 90% of 85% of some number, or if an amount is at 5% compounded yearly for two years, the successive multiplication model is quite appropriate. Many students think that if a person doubles his money and then triples what he has, he will wind up with five times the original sum. The model shows that reasoning to be inaccurate. Finally, the model has the advantage of being applicable to *all* real numbers and thus has advantages over repeated addition or subtraction models

or array models that often are used to represent multiplication.

Application 6: Addition of complex numbers is isomorphic to composition of two-dimensional translations (slides).

This isomorphism is used whenever the “parallelogram law” for addition of complex numbers is displayed. This picture of composition is the two-dimensional analogue of that used for real-number addition. In practicing complex-number addition, the following exercise might be more instructive: Given a figure graphed in the complex-number plane, what figure results from adding $2 - 3i$ to points in the original? (See fig. 6.)

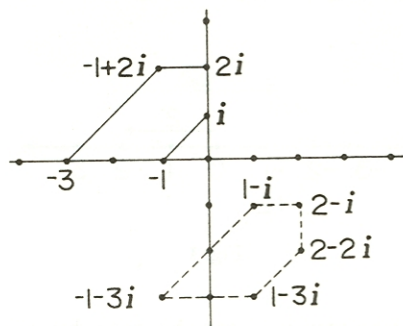


Fig. 6

Beginning at the origin and representing each of two slides by an arrow leads to the parallelogram law. The parallelogram law works for vector addition (where it originated) and indicates that vector addition is isomorphic both to the composition of translations and to complex-number addition.

A curious fact is that adding the complex number $a + bi$ results in the same operation that is used to graph $a + bi$ in rectangular coordinates, namely, sliding a units horizontally from $(0, 0)$ and then sliding b units vertically.

The fact is made all the more curious because multiplying by a complex number $a + bi$ results in the same operation that would be used to graph $a + bi$ in *polar coordinates* beginning at $(1, 0)$. That is, to plot $[r, \theta]$, one begins at $(1, 0)$, turns

θ , and goes to a distance r from the origin. ($[r, \theta] = r \text{ cis } \theta$ in more traditional but more unwieldy notation.) Similarly, if $a + bi = [r, \theta]$, then multiplying by $[r, \theta]$ will rotate a figure θ and size transform it by a factor of r . We exhibit this by multiplying the coordinates of a figure by $1 + i$. (See fig. 7.) Multiplying by $1 + i$ corresponds to rotating 45° and size transforming by a factor of $\sqrt{2}$. In polar coordinates, $1 + i = [\sqrt{2}, 45^\circ]$.

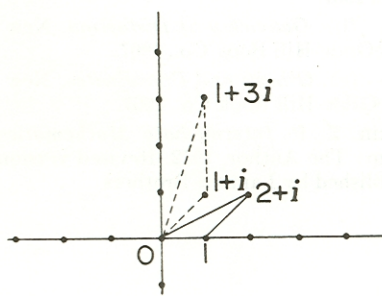


Fig. 7

The composition of a rotation and a size transformation with the same center is called a *spiral similarity* (see Coxeter 1961, pp. 72-75).

Application 7: Multiplication of complex numbers is isomorphic to composition of spiral similarities.

For example, multiplying by $2i$ rotates 90° and size transforms a magnitude 2 because $2i = [2, 90^\circ]$. Now we examine the multiplication $2i \cdot (-3i)$:

<i>Multiply</i>		<i>Compose</i>
$2i$		rotate 90°
	corresponds to	size transform 2
$-3i$		rotate 270°
		size transform 3
<hr style="width: 50px; margin: 0 auto;"/>		<hr style="width: 50px; margin: 0 auto;"/>
6		rotate 0°
		size transform 6

The answer 6 corresponds to a size transformation of magnitude 6, just as one would expect with this real number.

Some implications of this isomorphism are nontrivial. First, because of the con-

nection between spiral similarities and polar coordinates, it is easier to describe multiplication in polar coordinates than in either $a + bi$ or rectangular-coordinate notation. The general rules for multiplication in the three notations show the relative ease of polar coordinates:

$a + bi$ notation:
 $(a + bi)(c + di) = ac - bd + (ad + bc)i$

(a, b) notation:
 $(a, b)(c, d) = (ac - bd, ad + bc)$

$[r, \theta]$ notation:
 $[r, \theta] \cdot [s, \phi] = [rs, \theta + \phi]$

(An analogous situation with real numbers is that it is easier to multiply reals in fraction notation than in decimal notation.)

De Moivre's theorem is an immediate consequence of the polar-coordinate version of the multiplication rule. Again, the polar-coordinate notation is easier.

$a + bi$ notation:
 $(r(\cos \theta + i \sin \theta))^n = r^n (\cos n\theta + i \sin n\theta)$

$[r, \theta]$ notation:
 $[r, \theta]^n = [r^n, n\theta]$

The general message here is that when there exist isomorphic structures, it may be easier to think in one of the structures than the other. This is why we ask students to use arrows on a number line when they forget how to add positive and negative numbers. The geometrical version of complex-number multiplication helps to interpret $i^2 = -1$. Since $i = [1, 90^\circ]$, multiplication by i corresponds to rotating 90° . Two rotations of 90° give a rotation of 180° , which we identify with multiplication by the real number -1 .

Figure 8 shows a list of the largest additive and multiplicative groups of real and complex numbers and the isomorphic groups of transformations, serving as a summary of applications 4-7. The opera-

$\langle \text{set of real numbers, } + \rangle$	$\langle \text{set of one-dimensional translations, } \circ \rangle$
$\langle \text{set of nonzero real numbers, } \cdot \rangle$	$\langle \text{set of size transformations, } \circ \rangle$
$\langle \text{set of complex numbers, } + \rangle$	$\langle \text{set of two-dimensional translations, } \circ \rangle$
$\langle \text{set of nonzero complex numbers, } \cdot \rangle$	$\langle \text{set of spiral similarities, } \circ \rangle$

Fig. 8

tion of composition is denoted by the customary symbol \circ .

The second part of this article will appear in the next issue of the *MATHEMATICS TEACHER*. Contained therein are applications of groups and isomorphism to properties of powers, multiples, logarithms, congruence, similarity, symmetry, and the trigonometric functions.

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